Gravitational Instability in the Thermal Entry Region with Liquid Film Flow

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Thin film evaporators, vacuum distillation equipment, and other devices employed in chemical engineering processes involve the transport of heat through a flowing liquid film. When the flow is over a horizontal heated surface gravitational and/or Marangoni instabilities can occur to greatly increase the rate of transport processes through the liquid film. This note reports the results of a theoretical and experimental study of the onset of gravitational instability in a horizontally flowing liquid heated from below.

Since Rayleigh's classical analysis (1916) of the onset of convective motion in a stagnant film was published a considerable amount of work has been done to extend the analysis. The work can be classified as in the following diagram:

Rayleigh-like stability analyses

- Quiescent films
  - Linear temperature profile (A)
  - Nonlinear temperature profile (B)
- Flowing films
  - Linear temperature profile (C)

Considerable success has been realized in obtaining analytical understanding of the instability for case A. These classical results were summarized by Chandrasekhar (1961).

If the undisturbed temperature profile is nonlinear due to transient heating the critical Rayleigh number has been found to differ from the values associated with a linear temperature profile (Morton, 1957; Goldstein, 1959; Lick, 1965; and Currie, 1967). The quasi static or frozen time model used by these investigators was critically examined by Gresho and Sani (1971) who concluded that the quasi-static analysis, although valid asymptotically (at large times) is of limited usefulness in describing the overall stability of a transient system because in most cases the linearized equations will cease to be valid by the time that the asymptotic theory becomes valid. Foster (1965) examined the problem as an initial value problem, and by a similar approach Gresho and Sani found the critical Rayleigh number to be higher than that calculated from the frozen model. Unfortunately, their approach does not yield a unique solution for design purposes because of the arbitrariness of the initial conditions and the definition of the critical condition.

The stability criteria of flowing films subject to a linear temperature profile was studied extensively by Chandra (1938) experimentally and by Gallagher and Mercer (1965) and Deardorff (1965) theoretically. They concluded that the longitudinal roll was the preferred mode of convection and the critical Rayleigh number is independent of shear. Thus for case C the onset of instability is the same as for case A.

Recently Frisk and Davis (1972) observed that an instability of the Rayleigh type occurs when a thin film of liquid flows over a heated plate under the influence of a concurrent gas flow. Typical experimental results are shown in Figure 1, a plot of the Nusselt number for heat transfer from the wall versus the dimensionless distance from the point of the step increase in the wall heat flux. The solid line is the theoretical prediction for an undisturbed laminar film flow. The results indicate that for runs 9 and 316 instability occurred at some downstream distance beyond which the Nusselt number deviates substantially from the values predicted for stable flow. In the thermal entry region where instability occurs the temperature profile is nonlinear so that the stability analyses of case C above do not apply. Although the system is at steady state, the stability problem is analogous to the time-dependent problem because of the nonlinearity of the temperature distribution. The spatially-dependent problem is studied in the present paper to predict the stability characteristics.

PROBLEM FORMULATION

Consider the steady and fully developed laminar flow of a liquid layer over a rigid plate with a step change in the wall temperature (or wall heat flux) located at the point $x = 0$ as shown in Figure 2. The undisturbed veloc-

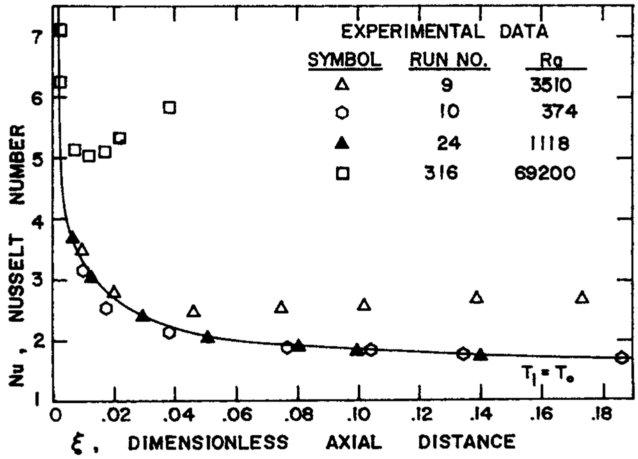

Fig. 1. Typical experimental results for stable and unstable films.

ity and temperature profiles are (in dimensionless form)

$$\overline{u} = \zeta \tag{1}$$

and

$$\overline{\theta} = \overline{\theta}_{fd} + \sum_{n=1}^{\infty} A_n Y_n(\zeta) \exp\left(-\lambda_n^2 \xi\right) \tag{2}$$

respectively, where $\overline{\theta}_{fd}$ is the fully developed temperature profile and the eigenfunctions $Y_n(\zeta)$ are given by

$$Y_n(\zeta) = K_1 \lambda_n^{1/3} \sqrt{\zeta}\, J_{1/3}\left(\frac{2}{3}\lambda_n \zeta^{3/2}\right) + k_2 \lambda_n^{1/3} \sqrt{\zeta}\, J_{-1/3}\left(\frac{2}{3}\lambda_n \zeta^{3/2}\right) \tag{3}$$

The coefficients K_1 and K_2 and the eigenvalues λ_n in Equation (3) are obtained from the boundary conditions listed in Table 1 and A_n, the eigenconstants, are obtained from the entry condition.

For case 1, a Leveque solution is more convenient for $\xi < 0.02$, that is,

$$\overline{\theta}(\xi, \zeta) = 1 - \frac{1}{0.893} \int_0^{\zeta/(9\xi)^{1/3}} e^{-\gamma^3}\, d\gamma \tag{4}$$

Equations (1), (2), or (4) represent the steady state undisturbed conditions.

To analyze the linear stability of the system we apply the usual method of introducing infinitesimal perturbations on the undisturbed components. Substituting the results in the equation of motion and the energy equation, applying the Boussinesq approximation, and neglecting viscous dissipation we obtain the following governing steady state perturbation equations in dimensionless form

$$\left(\frac{\zeta}{Pr}\frac{\partial}{\partial \xi} - \nabla^2_{\zeta,\eta}\right)\nabla^2_{\zeta,\eta}\, w^* - Ra\,\frac{\partial^2 \theta^*}{\partial \eta^2} = 0 \tag{5}$$

$$\left(\zeta \frac{\partial}{\partial \xi} - \nabla^2_{\zeta,\eta}\right)\theta^* + \left(u^* \frac{\partial}{\partial \xi} + w^* \frac{\partial}{\partial \zeta}\right)\overline{\theta} = 0 \tag{6}$$

$$\left(\frac{\zeta}{Pr}\frac{\partial}{\partial \xi} - \nabla^2_{\zeta,\eta}\right) u^* + \frac{1}{Pr} w^* = 0 \tag{7}$$

Introducing perturbations of the form

$$(w^*, \theta^*, u^*) \equiv (w^{**}, \theta^{**}, u^{**})(\zeta) \exp\left[i\,(k_\xi \xi + k_\eta \eta)\right] e^{\xi\sigma} \tag{8}$$

We reduce Equations (5) to (7) to

$$\left[\frac{\sigma}{Pr}\zeta - (D^2 - k_\eta^2)\right](D^2 - k_\eta^2)w + k_\eta^2\, Ra\theta = 0 \tag{9}$$

$$[\sigma\zeta - (D^2 - k_\eta^2)]\,\theta + \left[u\frac{\partial \overline{\theta}}{\partial \xi}(\xi, \zeta) + w\frac{\partial \overline{\theta}}{\partial \zeta}(\xi, \zeta)\right] = 0 \tag{10}$$

$$\left[\frac{\sigma}{Pr}\zeta - (D^2 - k_\eta^2)\right]u + \frac{1}{Pr}w = 0 \tag{11}$$

where $D \equiv d/d\zeta$, and $k_\xi = 0$ since the longitudinal roll is assumed to be the preferred mode of instability.

It is seen that the independent variables ξ and ζ are not separable because $\overline{\theta}$ is a nonlinear function of ξ as well as ζ. Since the variation of $\overline{\theta}$ with ξ becomes large near the leading edge of the heated surface, $\sigma = 0$ does not rigorously define the marginal state. By treating ξ as a parameter ξ^*, we can separate variables and let $\sigma = 0$ to study the stability of the system. This approximation is asymptotically correct because as ξ gets large $\overline{\theta}$ becomes a function of ζ only. For small ξ we cannot expect this analysis to be quantitatively valid, but the general stability characteristics can be obtained.

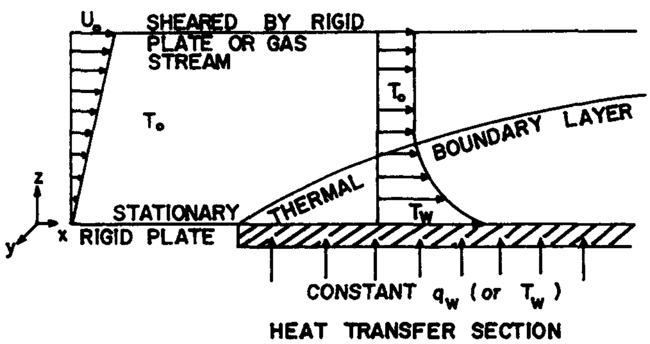

Fig. 2. A schematic diagram of the present problem.

TABLE 1. THERMAL BOUNDARY CONDITIONS

	Case 1	Case 2
at $z = 0$	T_w	q_w
at $z = \delta$	T_0	T_0
at $x = 0$	T_0	T_0

BOUNDARY CONDITIONS

The boundary conditions considered here for the perturbation components are tabulated in Table 2. For Set 1, experimental results are available.

SOLUTION OF EQUATIONS 9 TO 11

In the region $\xi \to \infty$, the undisturbed temperature profile is linear and ξ-independent, (that is, $\partial\overline{\theta}/\partial\zeta = -1$, $\partial\overline{\theta}/\partial\xi = 0$), the governing equations combine and reduce to

TABLE 2. BOUNDARY CONDITIONS FOR THE PERTURBATION COMPONENTS

Set	Boundary (R) or (F)	θ	$D\theta$	u	Du	w	Dw	D^2w	Remark
	Lower (R)	—	0	0	—	0	0	—	(R): Rigid
1	Upper (F)	0	—	—	0	0	—	0	(F): Free
	Lower (R)	0	—	0	—	0	0	—	Set 1: const. q_w and T_0
2	Upper (R)	0	—	0	—	0	0	—	Set 2: const. T_w and T_0
	Lower (R)	0	—	0	—	0	0	—	Set 3: const. T_w and T_0
3	Upper (F)	0	—	—	0	0	—	0	

$$(D^2 - k_\eta^2)^3 w + k_\eta^2 Ra\, w = 0 \qquad (12)$$

This is exactly the classical problem of the quiescent layer, so the classical solution describes this asymptotic region. For the thermal entry region Equations (9) to (11) may be written as a set of first-order differential equations which we solved by a fourth-order Runge-Kutta method. For given values of Pr and ξ^*, we obtain a curve Ra versus k_η which leads to nontrivial solutions. The minimum Ra of this curve is called the critical Rayleigh number Ra_c, and the corresponding wave number is called the critical wave number $k_{\eta,c}$.

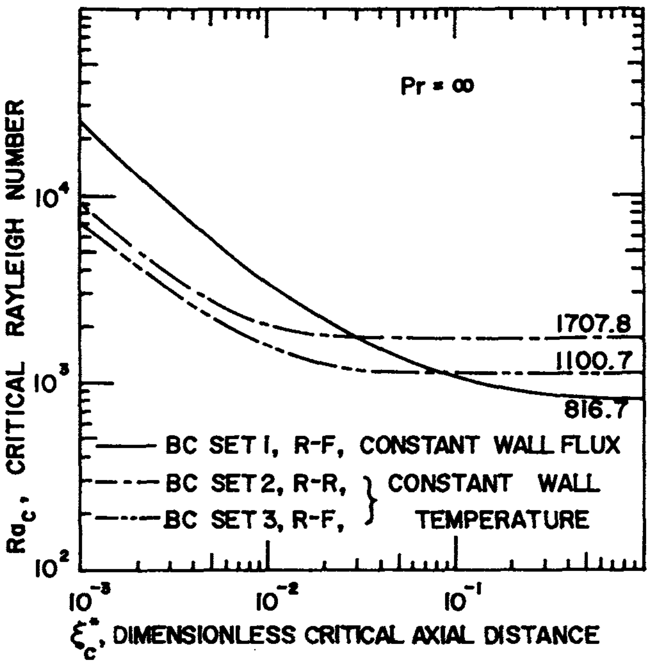

Fig. 3. Critical Rayleigh numbers for $Pr = \infty$.

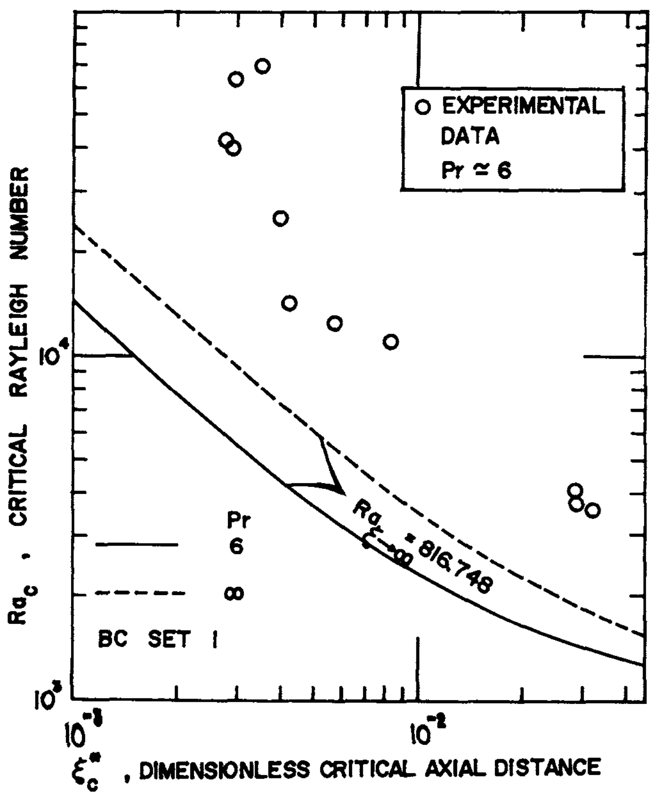

Fig. 4. The effect of the Prandtl number on the critical Rayleigh number.

RESULTS AND DISCUSSION

Typical results of Ra_c versus the critical position ξ_c^* subject to the boundary conditions of Table 2 are given in Figure 3. (Note that the characteristic temperature used in Ra is $q_w\delta/k$ for constant wall flux and $T_w - T_0$ for constant wall temperature.) As $\xi_c^* \to \infty$, one retains the classical solutions formulated for the quiescent layer. As ξ_c^* decreases, Ra_c increases, which implies that the liquid film is more stable with decreasing ξ_c^*. Since the undisturbed temperature profile is more nonlinear for smaller ξ_c^*, the nonlinearity of the undisturbed temperature profile becomes a stabilizing factor here. For $\xi = x/\delta RePr$, a given liquid (that is, given Pr) becomes more stable if we increase the rate of shear.

Figure 4 shows the effect of the Prandtl number on the stability for the boundary conditions of Set 1. A fluid with larger Prandtl number appears more stable in the thermal entrance region. The solutions shown on Figure 4 are also compared with experimental data taken in this laboratory, partly by Frisk and Davis (see Figure 1) and supplemented by more recent data of the authors. The quantitative discrepancy between the experimental data and the analysis is due to the assumption that $\theta(\xi, \zeta)$ may be considered frozen at its local value. This approximation may be relaxed by treating the problem as an inlet value problem which, however, required knowledge of the form of the perturbation at $\xi = 0$. This information is not usually available in applications.

NOTATION

A_n = eigenconstants
g = gravitational constant
K_1, K_2 = defined coefficients
k = thermal conductivity
k_ξ, k_η = dimensionless wave numbers of the disturbances in ξ and η directions
Nu = Nusselt number $(h\delta/k)$
Pr = Prandtl number (ν/κ)
p = dimensionless pressure
q = dimensional heat flux
Ra = Rayleigh number $(g\alpha\theta_{ch}\delta^3/\kappa\nu)$
Re = Reynolds number $(\overline{U}_0\delta/\nu)$
T = dimensional temperature
u, v, w = dimensionless velocity in x, y, z directions $(U/\overline{U}_0, V\delta/\kappa, W\delta/\kappa)$
U, V, W = dimensional velocity in x, y, z directions
x, y, z = dimensional axial, transverse, vertical coordinates
Y_n = eigenfunctions

Greek Letters

α = coefficient of thermal expansion
κ = thermal diffusivity
θ = dimensionless temperature $(T - T_0)/\theta_{ch}$
θ_{ch} = dimensional characteristic temperature
ξ, η, ζ = dimensionless axial, transverse, vertical coordinates $(x/\delta RePr, y/\delta, z/\delta)$
ξ^* = defined parameter
σ = spatial growth constant
δ = dimensional film thickness
ν = kinematic viscosity
λ_n = eigenvalues
$\nabla^2_{\zeta,\eta}$ = Laplace operator with respect to ζ and η

Subscripts

0 = inlet of heating
c = critical condition
w = wall, lower boundary

Superscripts

— = undisturbed phase
* = perturbed phase
** = amplitude of the disturbances

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Manuscript received June 21, 1973, and accepted September 19, 1973.

Pore Accessibility Applied to Correction of Errors in Mercury Porosimetry

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Characterization of the structure of porous materials is often necessary for an understanding of the properties and behavior of these materials in such varied applications as catalysis, fuel cells, secondary oil recovery, etc. Traditionally, porosity and pore size distributions have been important measures of pore structure conveniently determined by mercury penetration porosimetry. However, researchers have begun to inquire concerning the accuracy of mercury porosimetry, as well as possible alternate means for quantification of pore structure. This note presents a flexible, general method for the correlation and correction of distortions in porosimetry measurements and advances the concept of pore accessibility as an important characteristic of pore structure.

Mercury intrusion porosimetry inherently introduces errors into the determination of pore size distributions. It will not account for pores which can only be entered through pores or necks narrower than the pore itself. Intruded volumes of these pores are incorrectly assigned to the radii of smaller pores leading to these partially accessible pores. The extent of these errors appear to be dependent on sample size. A thin sample will have most of these larger pores easily accessible to the sample surface while a thick sample will have many large pores buried deep within the pore structure.

Meyer (1953) developed a probability-based method to correct errors introduced by these partially accessible pores. Kzenyhek (1963) expanded on the earlier work of Fatt (1956) in an effort to produce a more generalized correction method which took into account the size of the porous sample. The present work advances the approaches of Kzenyhek and Fatt such that distributions of pore length and the number of pore which connect at pore junctions are included in a generalized approach to correction of porosimeter data.

SYSTEM MODEL

Stochastic simulations of mercury intrusion porosimetry were performed in order to determine the sensitivity of errors in pore size distributions to: (1) shape of the actual pore size distribution, (2) sample size, (3) variations of pore length with pore radius, and (4) average number of pores which meet at a pore intersection. For simplicity, various beta distributions (Greenkorn and Kessler, 1969) were used to assess the influence of the shape of the pore size distribution. Pore lengths were related to pore radii by means of an inverse power relationship suggested by Fatt (1956):

$$l = l_0\, r^{-\alpha} \qquad \alpha \geqq 0$$